Some important definitions:

Point Set: Point sets are sets whose elements are points or vectors in E^n (n-dimensional vector space).

Example: (i) Linear combination of two variables x_1, x_2 i.e. $ax_1 + bx_2 = c$ in E^2 i.e. a set of those points (x_1, x_2) which satisfy $S_1 = \{(x_1, x_2) : ax_1 + bx_2 = c\}$

(ii) Set of points lying inside a circle of unit radius with centre at origin in E^2 i.e. $S_2 = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$

Line Segment: The line segment joining two points x_1, x_2 is defined to be the set L of points satisfying.

$$L = \{x : x = \lambda x_1 + (1-\lambda)x_2 : 0 \leq \lambda \leq 1\}$$

Hyperplane: A hyperplane is a set of points

$$x = \{x_1, x_2, \cdots, x_n\} \text{ satisfying}$$

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = z \quad (\text{not all } c_i = 0) \text{ i.e. } cx = z.$$

Parallel hyperplanes: Two hyperplanes $c_1x = z_1$ & $c_2x = z_2$ are said to be parallel if $c_1 = \lambda c_2$ for some $\lambda \neq 0$.

Open and Closed Half Spaces: A hyperplane divides the whole space E^n into three mutually disjoint sets:

$H_1 = \{x : cx > z\}$ +ve open half spaces

$H_2 = \{x : cx = z\}$ Hyper planes

$H_3 = \{x : cx < z\}$ −ve open half spaces

$H_1' = \{x : cx \geq z\}$ +ve closed half spaces

$H_3' = \{x : cx \leq z\}$ −ve closed half spaces

These are generated by hyperplane.

Polytope:- The intersection of finite number of closed half spaces is called a polytope.

Hyper Sphere:- A hypersphere in R^n with centre at 'a' and radius $r>0$ is defined to be set of points

$$S = \{x : |x - a| = r\}$$

Thus the equation of hypersphere in R^n is

$$\sum_{i=1}^{n} (x_i - a_i)^2 = r^2.$$

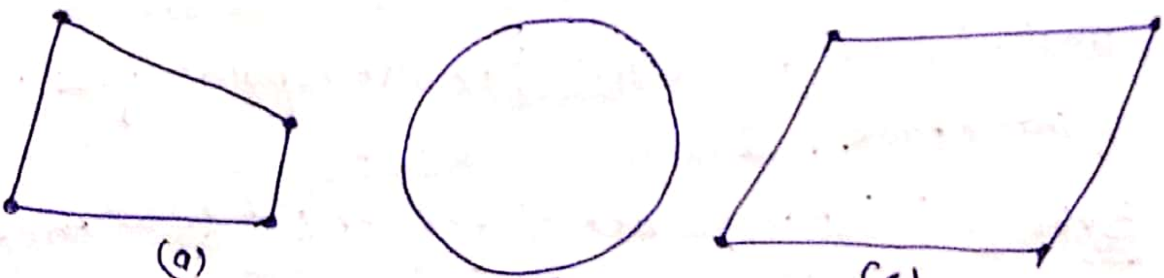
Convex Set:- A subset S of R^n is said to be convex if for any two points x_1 and x_2 in S, the line segment joining the points x_1 and x_2 is also contained in S.

In other words we can say that a subset S of R^n is said to be convex set iff $x_1, x_2 \in S$

$$\Rightarrow \{\lambda x_1 + (1-\lambda) x_2\} \in S \text{ for } 0 \le \lambda \le 1.$$

i.e. Convex combination of any two points in the set, is also in the set.

Some convex and Non convex sets in R^2 are given below:-



(a) (b) (c)

Fig: Convex Set

Fig: Non-convex set.

Example: Show that the set

$$S = \{(x_1, x_2) \mid 3x_1^2 + 2x_2^2 \le 6\}$$ is convex set.

Solution: Let x, y be two points of S.

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$

The line segment joining x and y is the set

$$\{u : u = \lambda x + (1-\lambda) y : 0 \le \lambda \le 1\}$$

For some λ, let $u = (u_1, u_2)$ be a point of this set,

so that $u_1 = \lambda x_1 + (1-\lambda) y_1$

& $u_2 = \lambda x_2 + (1-\lambda) y_2$

Now,

$$3u_1^2 + 2u_2^2 = 3\{\lambda x_1 + (1-\lambda) y_1\}^2 + 2\{\lambda x_2 + (1-\lambda) y_2\}^2$$

$$= (3x_1^2 + 2x_2^2)\lambda^2 + (3y_1^2 + 2y_2^2)(1-\lambda)^2 + (3x_1 y_1 + 2x_2 y_2) 2\lambda(1-\lambda)$$

$$\le 6\lambda^2 + 6(1-\lambda)^2 + 12\lambda(1-\lambda)$$

$$\because 3x_1 y_1 + 2x_2 y_2 \le \sqrt{(\sqrt{3}x_1)^2 + (\sqrt{2}x_2)^2} \cdot \sqrt{(\sqrt{3}y_1)^2 + (\sqrt{2}y_2)^2}$$

Thus, $3u_1^2 + 2u_2^2 \le 6$

and hence, $u = (u_1, u_2)$ is also a point of S.

Therefore, S is a convex set.

Extreme point or vertex of a convex set: An extreme point or vertex of a convex set is a point of the set which does not lie on any segment joining other two points of the set.

Thus, a point x of a convex set S is an extreme point of the set S if x cannot be expressed as linear combination of any two distinct points x_1 and x_2 in set S i.e.,

$$x \neq \lambda x_1 + (1-\lambda)x_2 \quad , \quad 0 < \lambda \leq 1$$

Note: (i) An extreme point is a boundary point of a set but converse is not true.

(ii) Extreme point cannot be between any two points of the set.

(iii) A singleton set has only one extreme point.

Convex combination:— A linear combination

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n \text{ is}$$

called a convex combination of given vector $x_1, x_2, \cdots x_n$ if each $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$.

Thus, a line segment joining two points x_1 & x_2 is a set of all convex combination of x_1 and x_2.

Theorem:— A hyperplane in R^n is a convex set.

Proof:— Let $cx = z$ be a hyperplane and also consider x_1, x_2 are any two points on the hyperplane X.

$$\therefore cx_1 = z \quad \text{and} \quad cx_2 = z$$

$$\text{If } x_3 = \lambda x_1 + (1-\lambda)x_2$$

then $\quad cx_3 = c\lambda x_1 + c(1-\lambda)x_2$

$$= \lambda z + (1-\lambda)z = z$$

$\Rightarrow x_3 = \lambda x_1 + (1-\lambda)x_2$ is also a point in the hyperplane X.

$\therefore$ By definition, the hyperplane X is a convex set.

Convex Polyhedron :— The set of all convex combination of a finite number of linearly independent vectors is called a convex polyhedron.

The convex polyhedron generated by the finite set of linearly independent vectors $x_1, x_2, \ldots, x_n$ is the set

$$\{x : x = \sum_{i=1}^{n} \lambda_i x_i \; ; \; \lambda_i \geq 0 ; \; \sum_{i=1}^{n} \lambda_i = 1\}$$

Convex Cone :— A non-empty subset C of R^n is called a cone if for each x in C and $\lambda \geq 0$, the vector λx is also in C.

A cone is called a convex cone if it is a convex set.

Convex hull of a set :— Let A is a set which is not convex. Then the smallest convex set which contains A is called the convex hull of A. i.e. the convex hull of a set A is the intersection of all convex set which contain A.

Example :— The convex hull of the set

$$A = \{(x, y) : x^2 + y^2 = 1\}$$

is the set $S = \{(x, y) \mid x^2 + y^2 \leq 1\}$

Here, S is a convex set and it contains A. Observe that the set S is smallest convex set containing A.

Theorem :— If A is any finite subset of vectors in R^n, then the convex hull of A is set of all convex combination of vectors in A.

Proof:- Let A be any finite sub-set of vectors in R^n, and $\langle A\rangle$ be its convex hull. Let S be the set of all convex combination of vectors in A.

We know that S is a convex set containing A and hence $\langle A\rangle \subset S$.

Also $\langle A\rangle$ containing A.

We shall show that $\langle A\rangle \supset S$.

To prove this result we use finite induction on the number of vectors in A.

For $k=2$. Let x_1, x_2 be two vectors in A. Then $\langle A\rangle$ contains x_1, x_2 and is a convex set.

$$\therefore x = \lambda x_1 + (1-\lambda)x_2 \; ; \quad 0 \le \lambda \le 1$$

is also in $\langle A\rangle$.

Hence, also all possible convex combination of x_1, x_2 are in $\langle A\rangle$.

Assume that for any +ve integer k, the result is true for a set of at most $k-1$ vectors.

Now, consider the set $A = \{x_1, x_2, \ldots, x_n\}$

Then $\langle A\rangle$ is convex set containing A.

$$\text{Let } X_i = \left\{x : x = \sum_{i=1}^{k-1} \mu_i x_i \; ; \; \mu_i \ge 0, \sum_{i=1}^{k-1} \mu_i = 1\right\}$$

Thus by induction hypothesis $\langle A\rangle \supset X$.

Now $\langle A\rangle$ contain X_i as well as $\{x_i\}$.

$\therefore$ A contains all segment joining x_k to a point of X_i.

$$\text{or} \quad x = \delta x_k + (1-\delta)\sum_{i=1}^{k-1} \mu_i x_i \; ; \mu_i \ge 0, \sum_{i=1}^{k-1} \mu_i = 1\}$$

is a point in $\langle A\rangle$.

$$\text{Now} \quad x = \delta x_k + (1-\delta)\sum_{i=1}^{k-1} \mu_i x_i$$

$\Rightarrow x = \sum_{i=1}^{k-1} \beta_i x_i$ for some $\beta_i = (1-\delta)\mu_i$

$i = 1, 2, 3, \ldots, k-1$. & $\beta_k = \delta$.

Since $\mu_i \geq 0$ for each i and $0 \leq \delta \leq 1$

$\therefore \beta_i \geq 0$

Also $\left\{ \sum_{j=1}^{n} \beta_j = \delta + (1-\delta) \sum_{i=1}^{k-1} \mu_i = 1 \right\}$

Hence, $x = \sum_{i=1}^{n} \beta_i x_i$ is a convex combination of the vectors $x_1, x_2, \ldots, x_k$ and is in $\langle A \rangle$.

i.e. $\langle A \rangle \supset S$.

Thus, we conclude that $\langle A \rangle = S$

In view of this theorem the convex polyhedron may be defined as, the convex hull of a set of points S consisting of a finite number of points in R^n.

Simplex:— A simplex is an n-dimensional convex polyhedron having exactly $(n+1)$ vertices.

Example:

(i) A simplex in zero dimension is a point.

(ii) A simplex in one dimension is a line segment.

(iii) A simplex in two dimension is a triangle.

(iv) A simplex in three (3) dimension is a tetrahedron and soon

Supporting Hyperplane:— Let $S \subseteq R^n$ be any closed convex set and $W \in S$ be a boundary point then a hyperplane $CX = Z$ is called a supporting hyperplane of S at W if

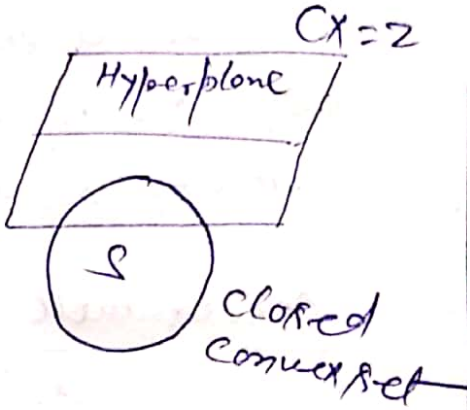
(I) $CW = Z$

(II) $S \subset H_+$ or $S \subset H_-$

Where

$H_+ = \{x : cx \geq z\}$

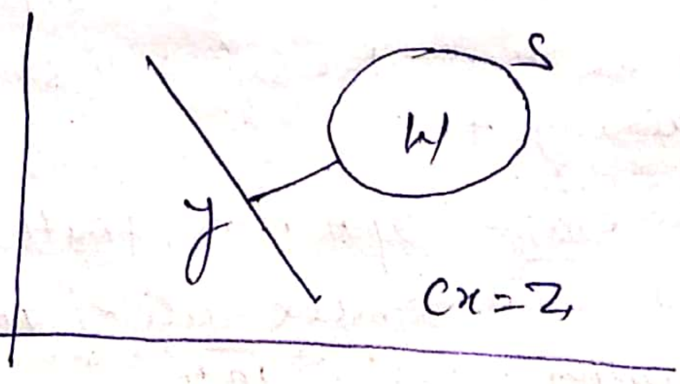
$H_- = \{x : cx \leq z\}$



Note: (I) Supporting hyperplane need not be unique.

(II) S may intersect the supporting hyperplane in more than one boundary point.

Separating Hyperplane:— Let $S \subset R^n$ be a closed convex set. Then for any point y not in S, there is a hyperplane s.t. S is contained in one of the open half spaces determined by the hyperplane.



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Theorem:- Let S be a closed convex set which is bounded below. Then S has extreme points in every supporting hyperplane.

Proof:- Let W be a boundary point of a closed convex set S.

Let $CX = Z$ be a supporting hyperplane at $W \in S$.

Let $B = S \cap \{x : Cx = Z\}$.

Then B is a closed convex set and $B \neq \phi$ for $W \in B$.

Here we claim that every extreme point of B is also an extreme point of S.

To get a contradiction let us assume that an extreme point b of B is not an extreme point of S.

Then $\exists$ $x_1, x_2 \in S$ s.t.

$$b = \lambda x_1 + (1-\lambda)x_2 \quad ; \quad 0 \leq \lambda \leq 1$$

$$\therefore \quad cb = c\lambda x_1 + c(1-\lambda)x_2 \quad \text{——— (I)}$$

Since $cx = z$ is a supporting hyperplane of W and $x_1, x_2 \in X$,

$$\left.\begin{array}{l} cx_1 \leq z \text{ and } cx_2 \leq z \\ cx_1 \geq z \text{ and } cx_2 \geq z \end{array}\right\} \quad \text{——— (II)}$$

from eqns (I) & (II); we get

$$cb \leq \lambda z + (1-\lambda)z = z$$

or $$cb \geq \lambda z + (1-\lambda)z = z$$

Hence, b is not a point of B.

which is a contradiction.

Hence, every extreme point of B is also an extreme point of S.

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Convex Function:

Let S be a non-empty convex subset of R^n. A function $f(x)$ on S is said to be convex function if for any two vectors x_1 and x_2 in S

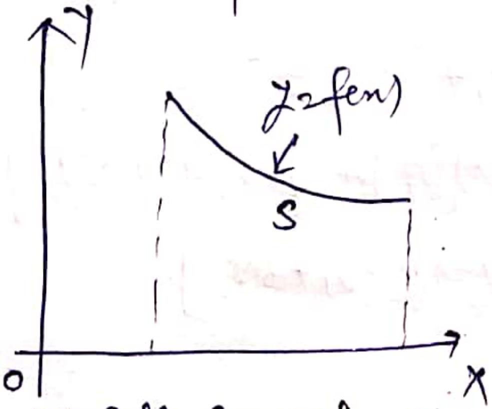
$$f\{\lambda x_1 + (1-\lambda) x_2\} \leq \lambda f(x_1) + (1-\lambda) f(x_2) \; ; \; 0 \leq \lambda \leq 1.$$

Strictly Convex Function: Let S be a non-empty convex subset of R^n. A function $f(x)$ on S is said to be strictly convex function if for any two different vectors x_1 and x_2 in S:

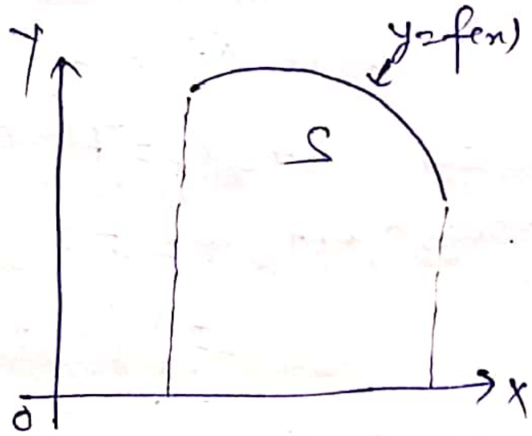
$$f\{\lambda x_1 + (1-\lambda) x_2\} < \lambda f(x_1) + (1-\lambda) f(x_2) \, , \; 0 < \lambda < 1.$$

Concave Function: A function $f(x)$ on a non-empty subset of R^n is said to be concave function if $-f(x)$ is a convex function.

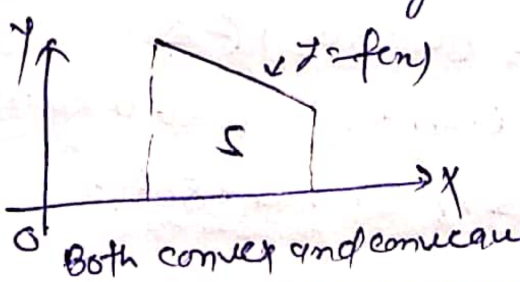
Similarly, the function $f(x)$ is said to be strictly concave function if $-f(x)$ is strictly convex function.



strictly convex function



strictly concave function



Both convex and concave function.

Note: (i) Every strictly convex function is also convex.

(ii) It is possible for a function to be both convex and concave. For example $y = f(x) = x$ is such a function.

(iii) A linear function is a convex (concave) but can not be strictly convex (concave).

(iv) The sum of two convex (concave) function is convex (concave) and if at least one of the function is strictly convex (concave) then their sum is so.

Example: Show that $S = \{(x_1, x_2) \in R^2 / 2x_1 + 3x_2 = 7\}$ is is a convex set.

Solution: Let $X, Y \in S$, where $X = (x_1, x_2)$, $Y = (y_1, y_2)$

The line segment joining X and Y is the set

$$W = \{W : W = \lambda X + (1-\lambda) Y : 0 \le \lambda \le 1\}$$

Let $W = (\omega_1, \omega_2)$ be a point of set W, so that

$$\omega_1 = \lambda x_1 + (1-\lambda) y_1 ; \quad \omega_2 = \lambda x_2 + (1-\lambda) y_2$$

Since $X, Y \in S$.

$$\Rightarrow 2x_1 + 3x_2 = 7 \quad \text{and} \quad 2y_1 + 3y_2 = 7$$

Now,
$$\begin{aligned} 2\omega_1 + 3\omega_2 &= 2[\lambda x_1 + (1-\lambda) y_1] + 3[\lambda x_2 + (1-\lambda) y_2] \\ &= \lambda [2x_1 + 3x_2] + (1-\lambda)[2y_1 + 3y_2] \\ &= \lambda \cdot 7 + (1-\lambda) 7 \\ &= 7 \end{aligned}$$

Hence $W = (\omega_1, \omega_2) \in S$

i.e. $W = (\omega_1, \omega_2)$ is also a point of S.

Hence S is a convex set.

Example: Show that $S = \{(x_1, x_2) \in R^2 \mid x_1 + 2x_2 = 3\}$ is convex set.

Solution: Let $X, Y \in S$

where, $X = (x_1, x_2)$, $Y = (y_1, y_2)$

The line segment joining X and Y is the set

$W = \{W : W = \lambda X + (1-\lambda) Y : 0 \le \lambda \le 1\}$

For some λ, $0 \le \lambda \le 1$, let $W = (w_1, w_2)$ be a point of set W, so that

$w_1 = \lambda x_1 + (1-\lambda) y_1$; $w_2 = \lambda x_2 + (1-\lambda) y_2$

$\because X, Y \in S$

$\Rightarrow x_1 + 2x_2 = 3$, $y_1 + 2y_2 = 3$

Now $w_1 + 2w_2 = [\lambda x_1 + (1-\lambda) y_1] + 2[\lambda x_2 + (1-\lambda) y_2]$

$= \lambda(x_1 + 2x_2) + (1-\lambda)(y_1 + 2y_2)$

$= 3\lambda + (1-\lambda)3$

$= 3$

Hence, $W = (w_1, w_2)$ is also a point of S,

Hence, S is a convex set